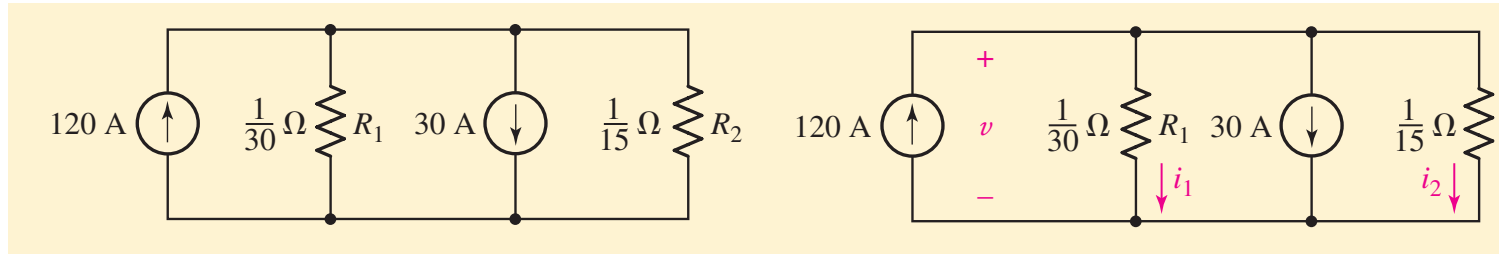


Linear algebra for data science

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Recall Basic Electrical Engineering



Find currents and voltages

Electrical Networks

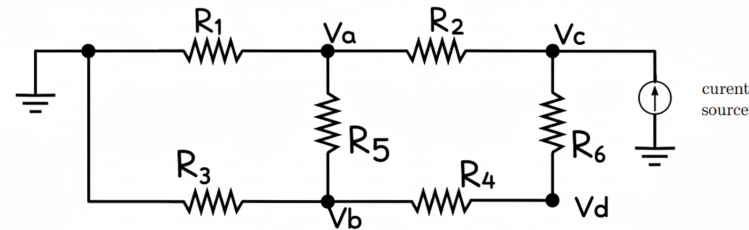


Figure: A resistive network.

In conductance form ($G_i = 1/R_i$) these can be written compactly as

$$\begin{aligned} (G_1 + G_2 + G_5)V_a - G_5V_b - G_2V_c &= 0, \\ -G_5V_a + (G_3 + G_4 + G_5)V_b - G_4V_d &= 0, \\ -G_2V_a + (G_2 + G_6)V_c - G_6V_d &= I_s, \\ -G_4V_b - G_6V_c + (G_4 + G_6)V_d &= 0. \end{aligned}$$

Thus the matrix equation $A\mathbf{v} = \mathbf{b}$ is

$$\begin{pmatrix} G_1 + G_2 + G_5 & -G_5 & -G_2 & 0 \\ -G_5 & G_3 + G_4 + G_5 & 0 & -G_4 \\ -G_2 & 0 & G_2 + G_6 & -G_6 \\ 0 & -G_4 & -G_6 & G_4 + G_6 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_s \\ 0 \end{pmatrix}.$$

Counting

- In how many ways can Roger and Rafa win 10 grand slams between themselves

$$R_o + R_a = 10$$

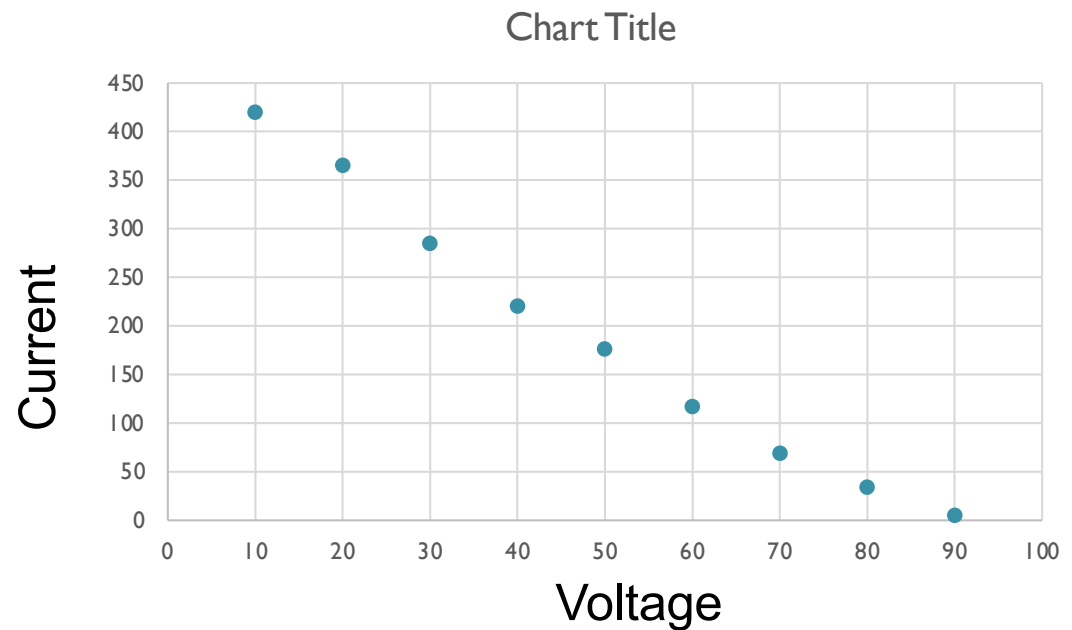
- In how many ways can write 10 as a sum of two 'real' numbers

$$x_1 + x_2 = 10$$

A simple college experiment

Given an unknown resistance? How would you find its value?
You have a voltage source and a current measuring device

Voltage	Current
10	50
20	100
30	150
40	200
50	250
60	300
70	350
80	400
90	450



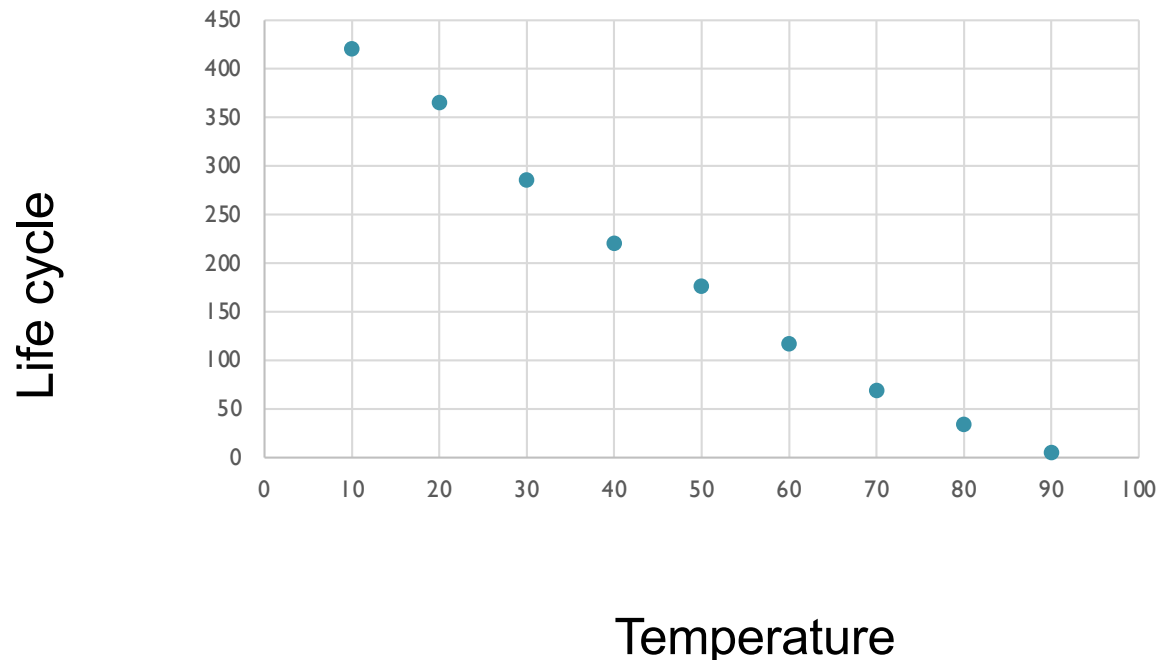
Prediction?

A company manufactures an electronic device to be used in a very wide temperature range. The company knows that increased temperature shortens the life time of the device, and a study is therefore performed in which the life time is determined as a function of temperature. The following data is found:

What is the life for 65 degrees?

Temperature in Celcius (t)	10	20	30	40	50	60	70	80	90
Life time in hours (y)	420	365	285	220	176	117	69	34	5

Chart Title



Contents

1	Geometry of linear equations
	Row picture
	Column picture
	Matrix form
2	Types of solutions
	For 2–dimensional and 3–dimensional case
3	Vector spaces
	Vector
	Vector addition
	Vector space
	Linear combination
	Subspaces – row space, column space, null space
	Independence and dependence of equations
	Basis and dimension
	Matrix multiplication
	Symmetric matrix

Contents

4	Solving a system of linear equations - Gaussian elimination
	Augmented matrix and pivots
	Manipulating rules
	Step by step process of elimination
	Forward elimination and back substitution
	Reduced row echelon form (rref)
	When could the process breakdown?
	Finding inverses by Gauss–Jordan method
5	Orthogonality
	Length of a vector
	Test for orthogonality
	Orthonormal vectors
	Projection
	Gram–Schmidt orthogonalization

Contents

6	Eigenvalues and eigenvectors
	Finding eigenvectors for distinct eigenvalues
	Finding eigenvectors for repeated eigenvalues
	Diagonalization of a symmetric matrix – Eigenvalue decomposition

Outcome

Module learning outcomes:

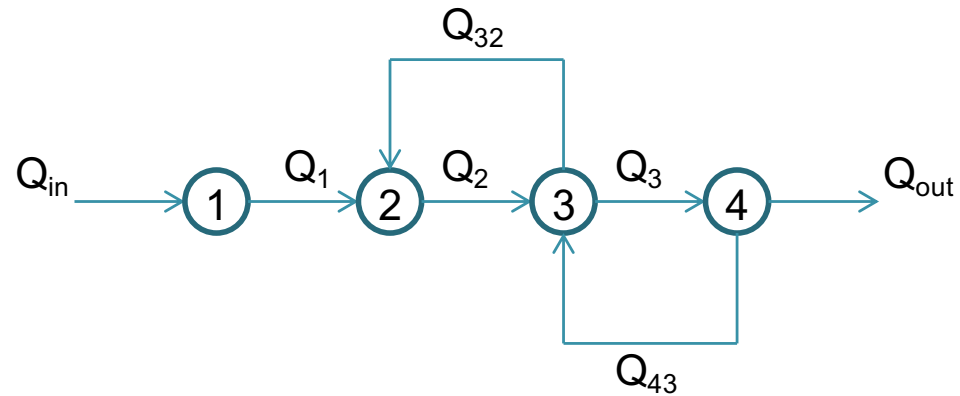
1. Participants will be able to identify relationships between variables in large datasets
2. Participants will be able to identify information sufficiency in terms of both equations and variables
3. Participants will be able to understand basic linear algebra concepts that underlie the complicated data analytics algorithms



Linear Algebra

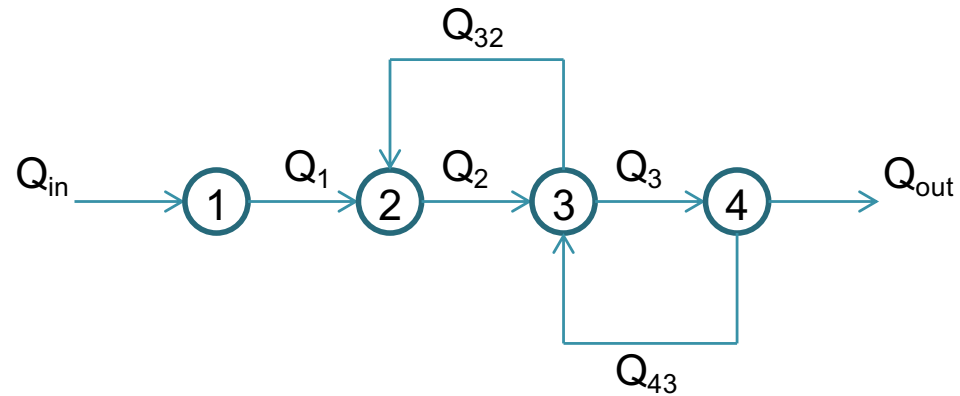


System of Equations



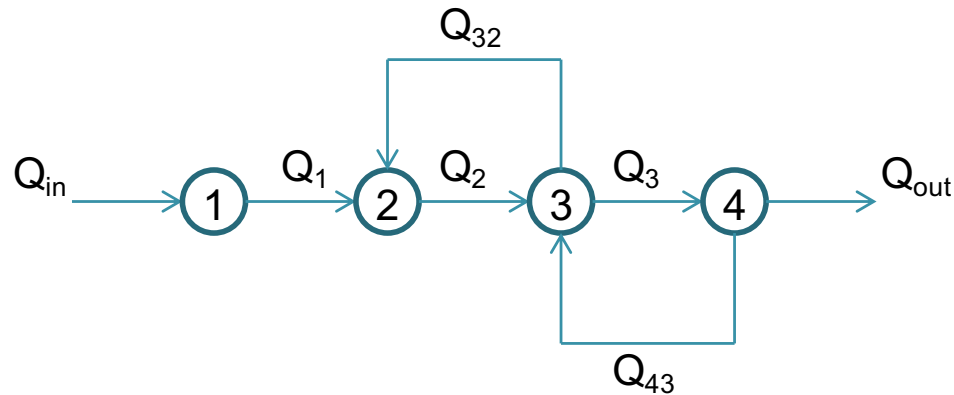
How many equations could be formed from this?

System of Equations



- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

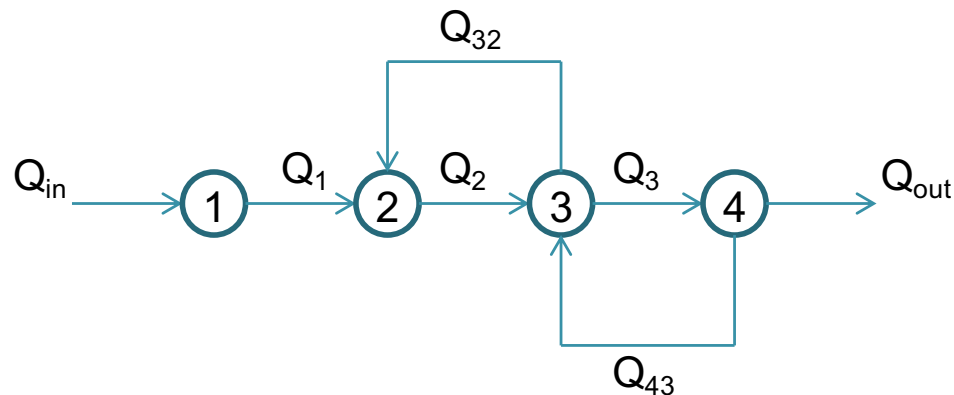
System of Equations



- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when –
 - $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$
 - $Q_{in} = 10$, $Q_{out} = 10$ and $Q_3 = 7$
 - $Q_{in} = 10$, $Q_{out} = 11$ and $Q_3 = 7$

System of Equations

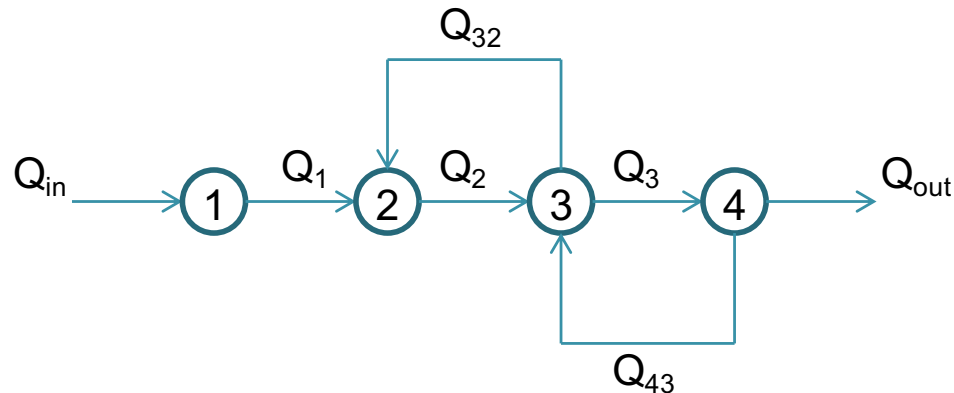


- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$?

Unique solution

System of Equations



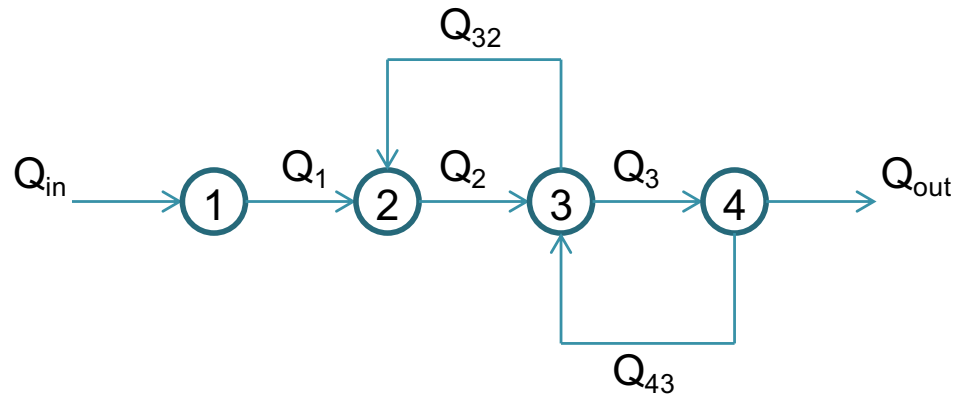
- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{out} = 10$ and $Q_3 = 17$?

Infinite number of solutions

- Loop involving Q_2 and Q_{32} : both are unmeasured
- Given measurements consistent with equations

System of Equations



- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{out} = 11$ and $Q_3 = 7$?

No solution

- Given information not consistent with equations

System of Equations: Key Concept

Understanding when do we have:

- Unique solution
- No solution
- Infinite number of solutions

Solving Simultaneous Linear Equations

Solve the two linear equations:

$$4x - 2y = 0 ; -2x + 4y = 6$$

Solving Simultaneous Linear Equations

Elimination (High-school method)

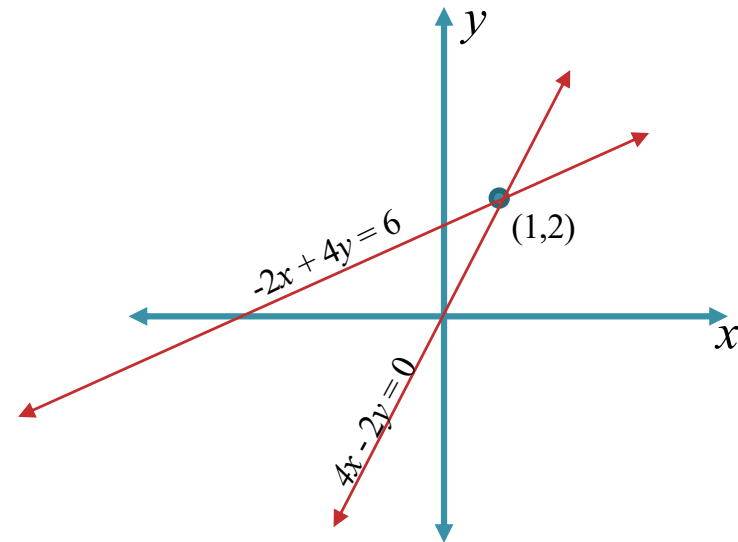
$$\begin{array}{r} 4x - 2y = 0 \\ 2(-2x + 4y = 6) \\ \hline 6y = 12 \end{array}$$

$$\Rightarrow y = 2$$

$$\Rightarrow x = 1$$

(1,2) is the solution

Graphical method



(1,2) is the solution

Geometry of Linear Equations

- We may view a system of linear equations in three different ways –
 - Matrix form – $\mathbf{Ax} = \mathbf{b}$ where \mathbf{A} forms a matrix with the coefficients of the unknowns and \mathbf{x} forms a matrix with the unknowns and \mathbf{b} , a matrix with the values in the R.H.S
 - Row picture – viewing one equation at a time
 - Column picture – two separate equations as one vector equation

Geometry of Linear Equations

Geometry for a system of 2 equations:

Matrix form

- Consider two linear equations:

$$4x - 2y = 0$$

$$-2x + 4y = 6$$

- A matrix is a rectangular arrangement of numbers in rows and columns
- Rows run horizontally and columns run vertically
- Order of a matrix: $m \times n$ where m is the # of rows and n is the # of columns

Geometry of Linear Equations

Matrix form

$$\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

- This is of the form $\mathbf{Ax} = \mathbf{b}$
where –
 - \mathbf{A} – matrix with coefficients of the unknowns
 - \mathbf{x} – unknowns
 - \mathbf{b} – R.H.S of the equations
- n equations and n unknowns $\Rightarrow n \times n$ matrix (square matrix)
- m equations and n unknowns $\Rightarrow m \times n$ matrix (rectangular matrix)

Geometry of Linear Equations

Row picture

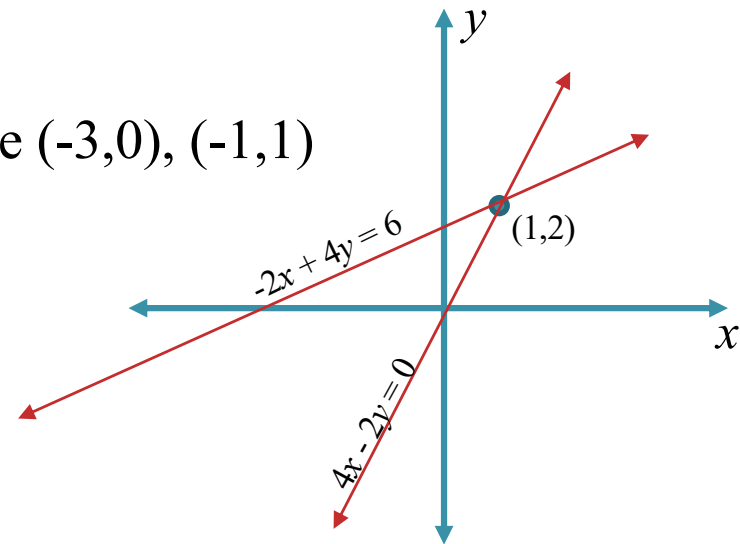
$$4x - 2y = 0 ; -2x + 4y = 6$$

- Taking one row at a time and plotting it in the x - y plane

Few points that satisfy $4x - 2y = 0$ are $(0,0)$, $(1,2)$
and $(1/2, 1)$

Few points that satisfy $-2x + 4y = 6$ are $(-3,0)$, $(-1,1)$
and $(1,2)$

So the solution of the system is $(1,2)$



Geometry of Linear Equations

Geometry for a system of 3 equations

- Consider a system of 3 equations:

$$x + 2y + z = 6$$

$$6x - 2y = 4$$

$$-3x - y + 4z = 8$$

Matrix form

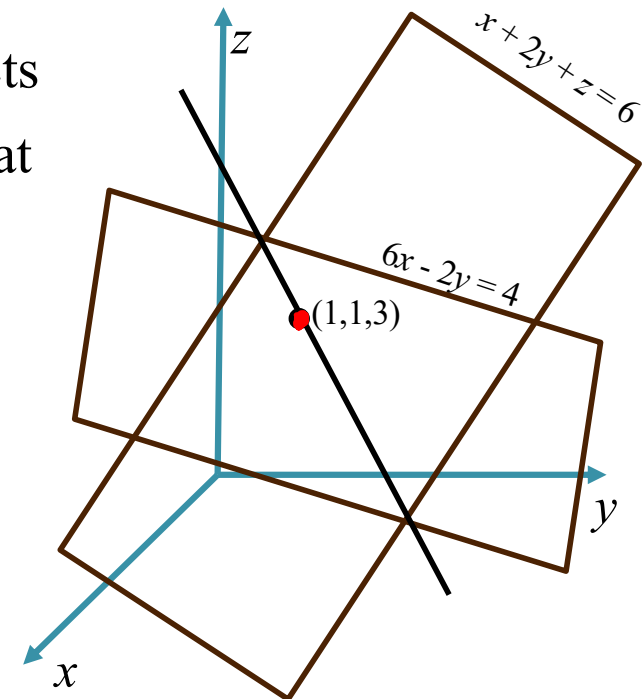
$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -2 & 0 \\ -3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}$$

Geometry of Linear Equations

Row picture

$$x + 2y + z = 6 ; 6x - 2y = 4 ; -3x - y + 4z = 8$$

- Each equation describes a plane in 3 dimensions. The intersection of the first plane with the second plane is a line
- The 3rd plane (not shown in the figure) intersects the line of intersection of the other two planes at a point $(1,1,3)$
- Solution for the system of equation is $(1,1,3)$



Types of Solutions

A system of linear equations can have –

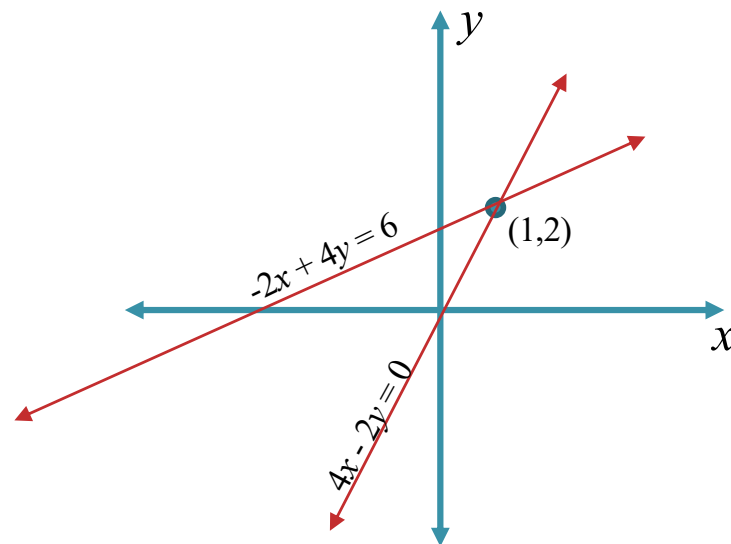
- Unique solution
or
- No solution
or
- Infinite number of solutions

Types of Solutions

2 dimensional case

Unique solution

- This is the nice case where the system will have a point of intersection and hence a unique solution. $4x - 2y = 0$ and $-2x + 4y = 6$ has a unique solution $(1,2)$



One solution $(x, y) = (1,2)$

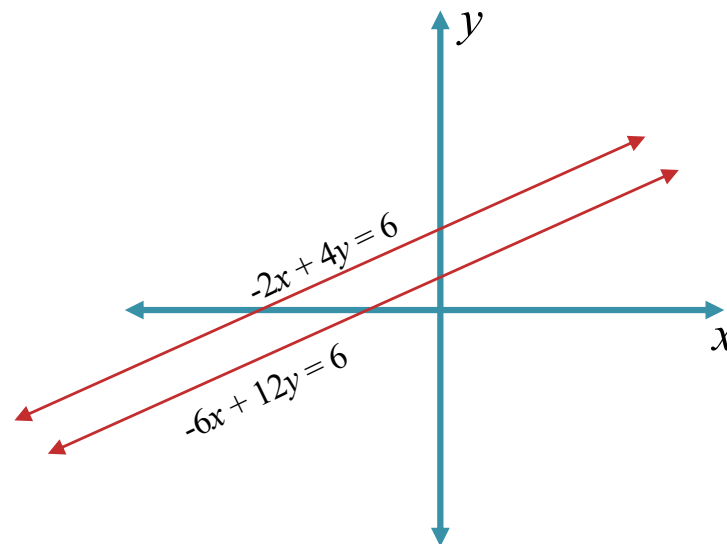
Types of Solutions

No solution

- A system has no solution if the equations are inconsistent. For example, $-2x + 4y = 6$ and $-6x + 12y = 6$ has no solution

$$-6x + 12y = 6 \Rightarrow -2x + 4y = 2$$

which contradicts with the first equation and hence the system has no solution



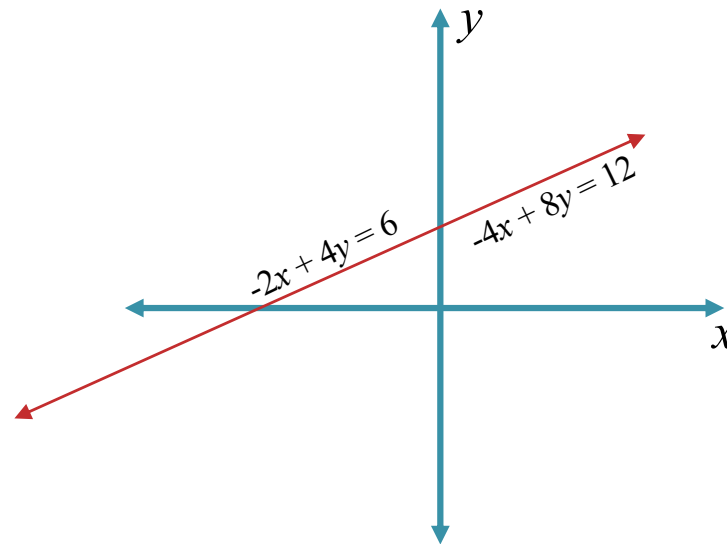
Parallel: No solution

Types of Solutions

Infinite number of solutions

- The other case is when one equation is just some multiple of the other. Then we will get infinite number of solutions

$$-2x + 4y = 6 ; -4x + 8y = 12$$

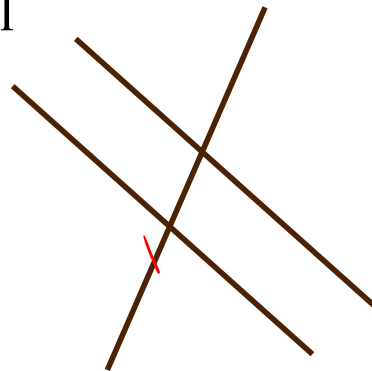


Whole line of solutions

Types of Solutions

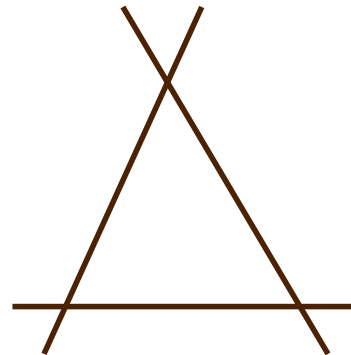
Singular case for three dimensions

- Two planes may be parallel



Two parallel planes: No solution

- One plane is parallel to the line of intersection of the other two planes

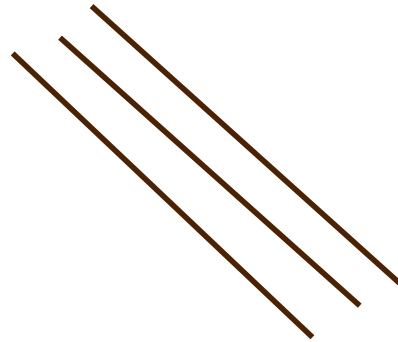


No intersection: No solution

Types of Solutions

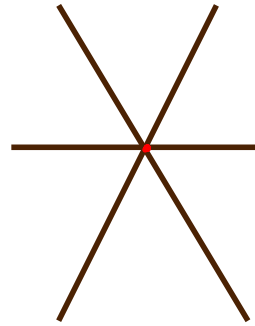
Singular case for three dimensions

- Three parallel planes



All planes parallel: No solution or a whole plane of solutions

- One equation is just the sum of the other two equations, the three planes have a whole line in common



**Line of intersection:
Infinite # of solutions**

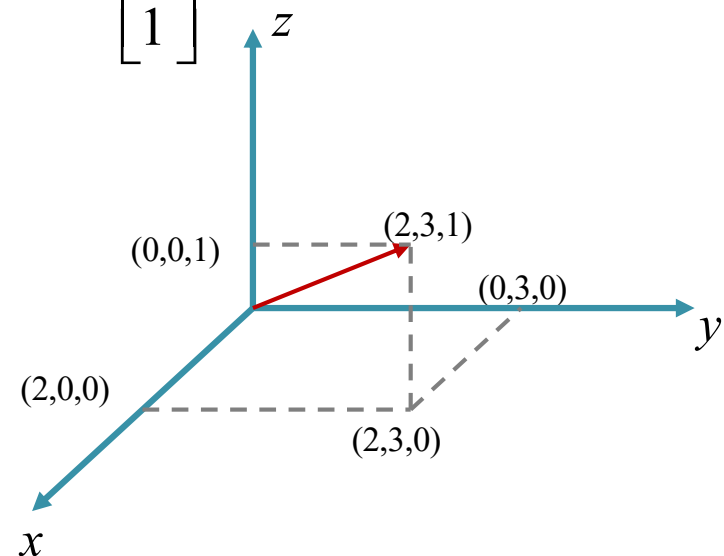
Geometry of Linear Equations

Vector

- A vector is defined as an ordered collection of numbers
- Elements of a vector arranged as a column \rightarrow column vector
- Elements of a vector arranged as a row \rightarrow row vector

- If a vector v contains three real numbers say, $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, then v belongs to the vector space \mathbb{R}^3

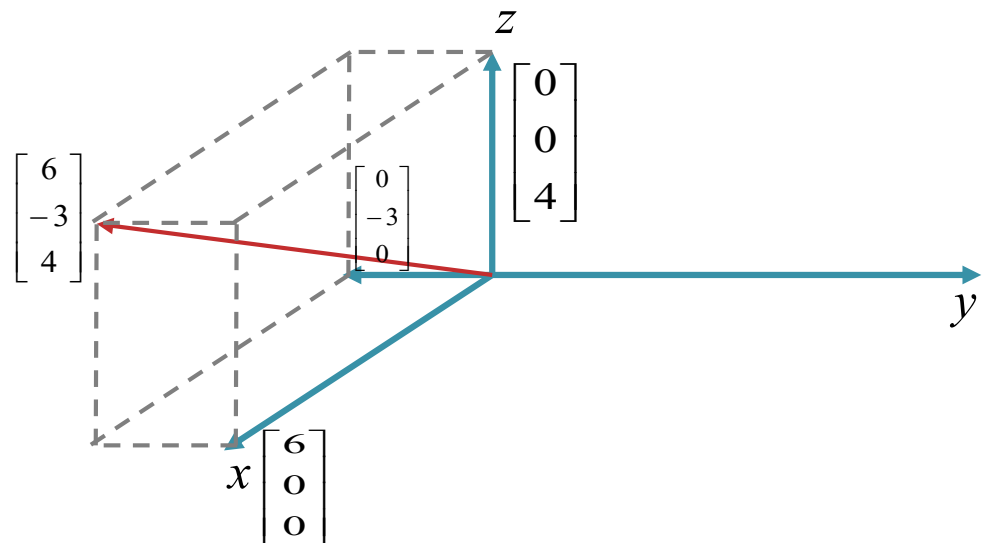
- The vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ a \end{bmatrix}$ are not the same



Vector Addition

Addition of a vector $\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$ is done component by component and can be written as –

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$$



Geometry of Linear Equations

Column picture

$$4x - 2y = 0 \quad ; \quad -2x + 4y = 6$$

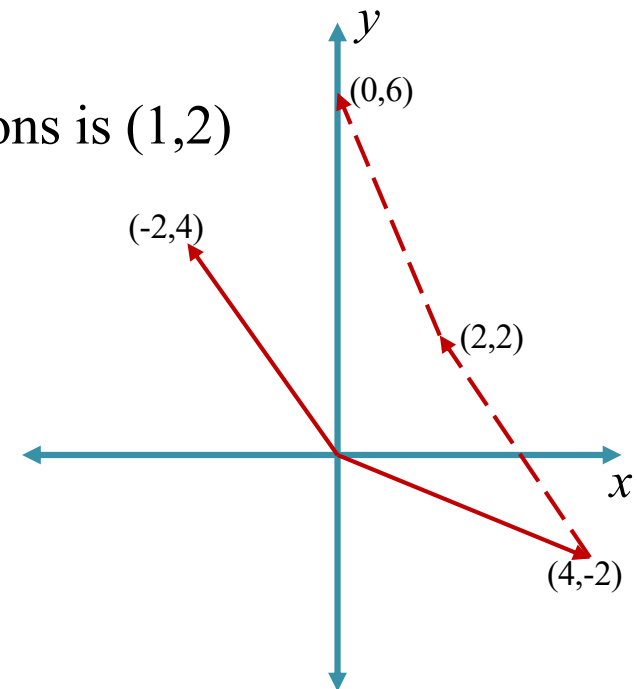
- Column picture: Linear combination of columns

$$x \begin{bmatrix} 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

We know that the solution for the two equations is (1,2)

Substitute them

$$1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



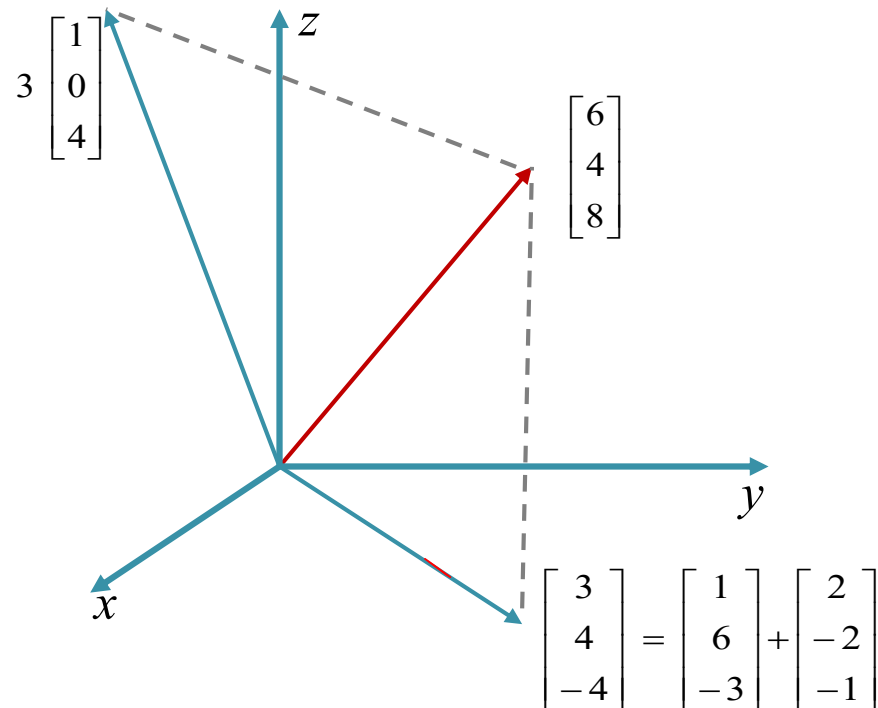
Geometry of Linear Equations

Column picture

$$x + 2y + z = 6 ; 6x - 2y = 4 ; -3x - y + 4z = 8$$

$$x \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} + y \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}$$

Solution for the system of equation is
(1,1,3)

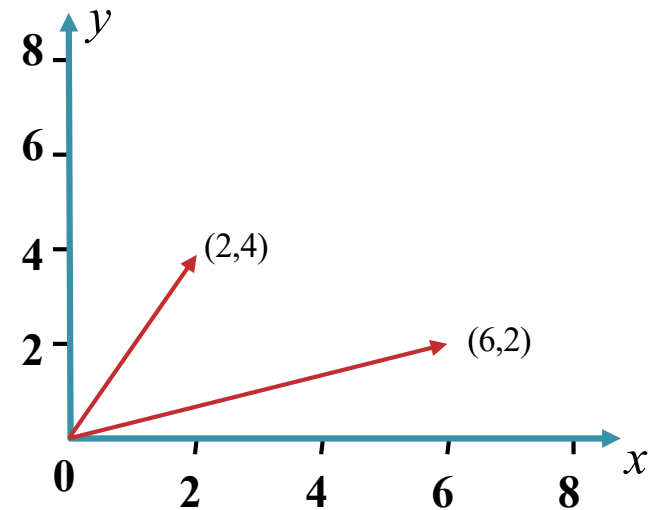


Vector Space

- Let V be a set of all vectors that lie in the first quadrant of R^2 and F be R

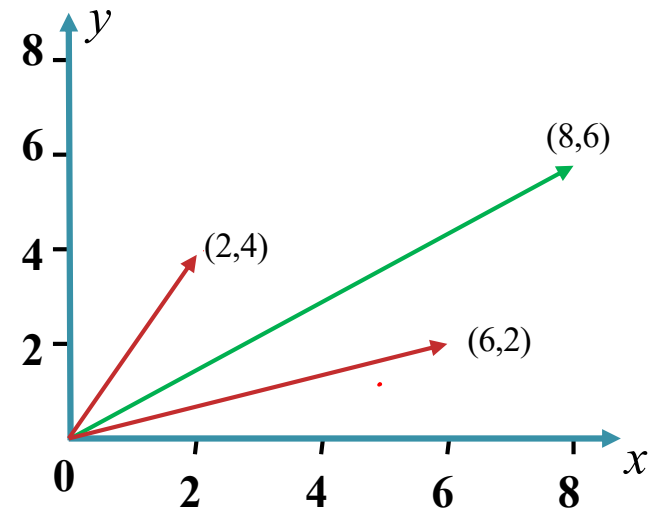
- Consider two vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 2 \end{bmatrix} \in V$

- It can easily be noted that both these vectors lie in the first quadrant of R^2



Vector Space

- Addition: $\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \in V$
- The resulting vector also lies in the first quadrant of \mathbb{R}^2



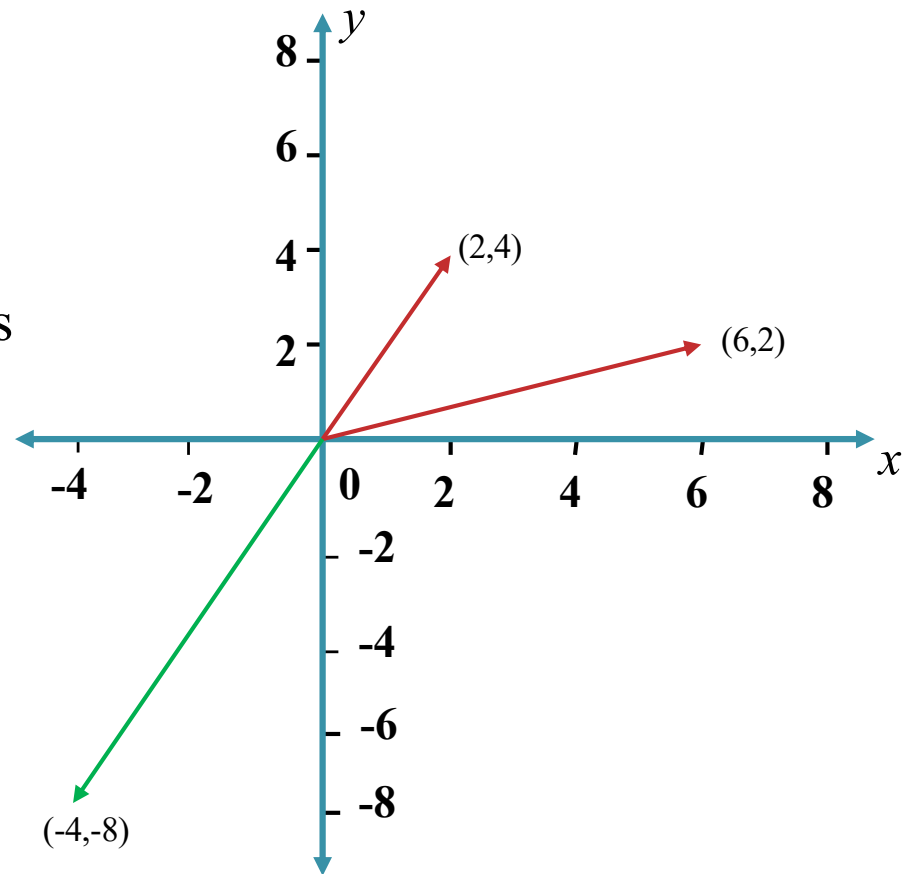
Vector Space

- Scalar multiplication: $a \in F$ where F is a field R .

Consider $a = -2$

$$\text{Then, } -2 \times \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$$

- It is clear that the resulting vector is outside V and hence it is not a vector space
- So, the first quadrant of R^2 is not a vector space whereas, R^2 is a vector space



Vector Space

- Let V be a set of vectors and F be a field. Then V is called a vector space over a field F if the following requirements are met

$$\forall x, y \in V, (x + y) \in V$$

$$\forall x \in V \text{ and } \forall a \in F, a \times x \in V$$

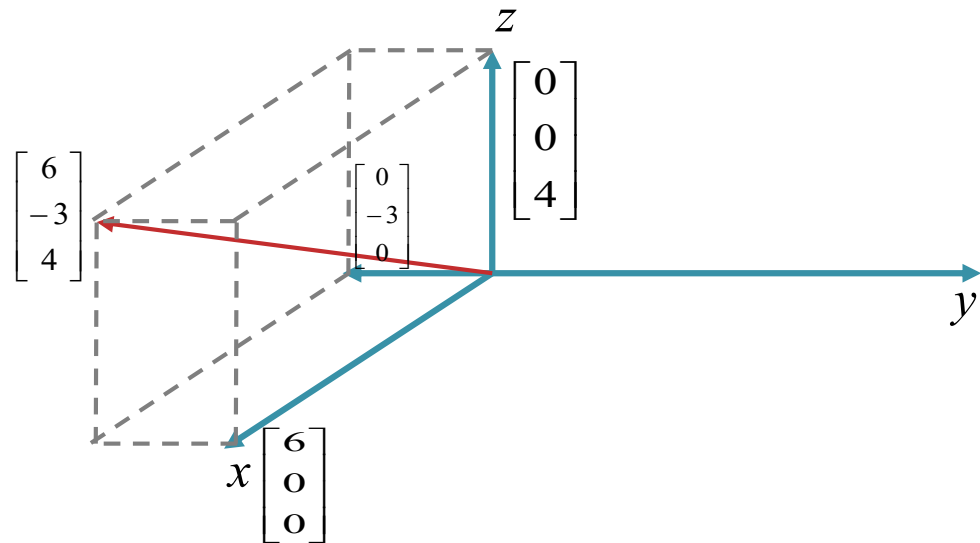
- In other words, a set of vectors is called a vector space if the set is closed under vector addition and scalar multiplication of a vector

Linear Combination

- The vector $\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$ can be expressed as a linear combination of vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ as below}$$

$$\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Linear Combination

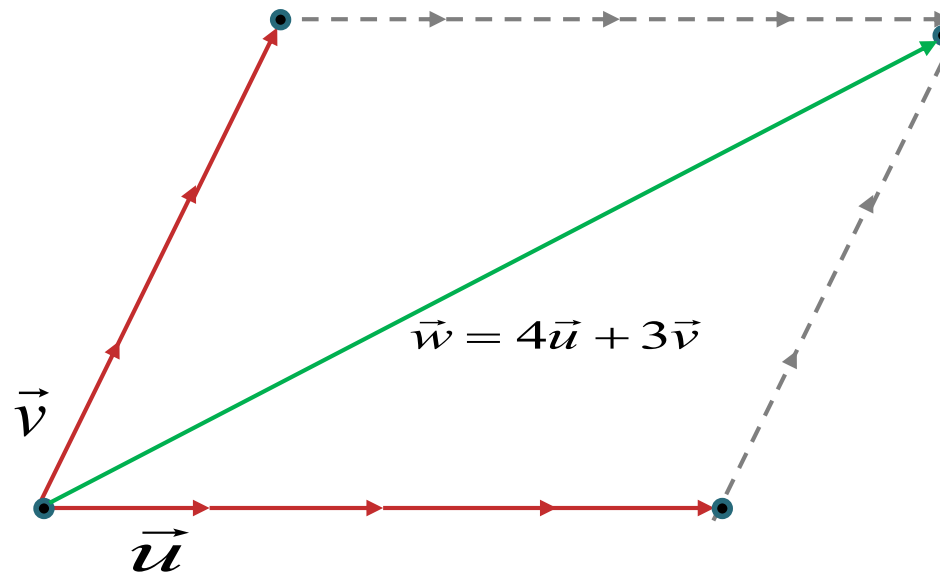
- A vector v can be written as a **linear combination** of vectors $u_1, u_2, u_3, \dots, u_n$ such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

where c_1, c_2, \dots, c_n are all scalars

Linear Combination

- Geometrically, we may see the linear combination as –



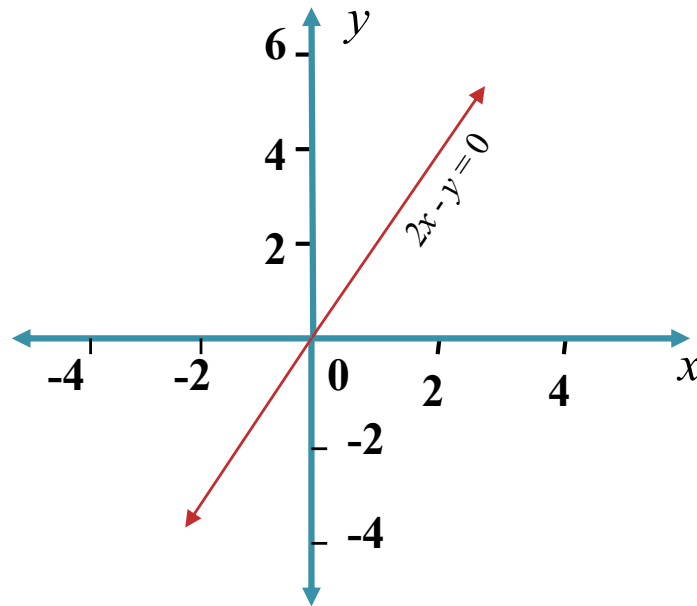
- We have the vector \vec{w} which is expressed as a linear combination of the other two vectors \vec{u} and \vec{v}

Subspace

- R^2 satisfies the conditions for a vector space – vector addition and scalar multiplication
- What about the line $2x - y = 0$? Does it satisfy the conditions?

Answer: Yes

Any line which passes through the origin in R^2 is a subspace of R^2



Subspace

- A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space –
 - Vector addition: For any vectors x, y in the subspace, $x + y$ is in the subspace
 - Scalar multiplication: For any scalar c , cx is in the subspace
- Column space: Contains all linear combinations of the columns of \mathbf{A}
- Row space: Contains all linear combinations of the rows of \mathbf{A}

Subspace

Is column space a subspace?

- Column space contains all linear combinations of the columns of **A**

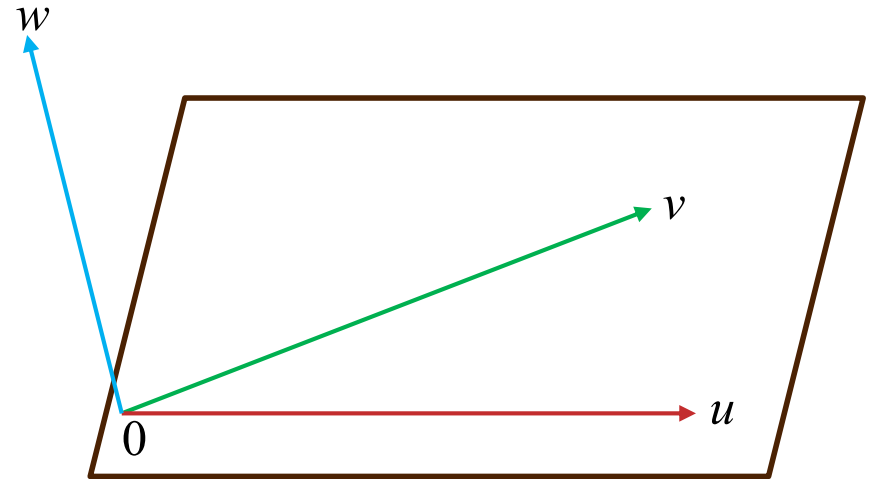
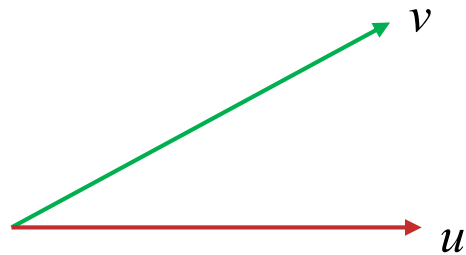
- $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$; Column space of A contains all linear combinations of

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

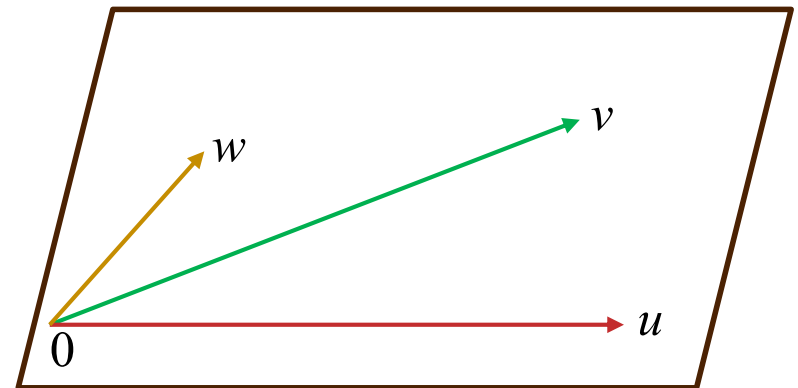
- If we take all the linear combinations of the columns in A, it will fill a plane in $\mathbb{R}^3 \Rightarrow$ it is a subspace of \mathbb{R}^3

Linear Dependence & Independence

- Linearly independent:



- Linearly dependent:



Linear Independence

- A set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ is said to be **linearly independent** if $c_1 = c_2 = c_3 = \dots = c_n = 0$ is the only solution to the following equation

$$c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_nv_n = 0$$

- Any one vector in the set cannot be expressed as a linear combinations of the rest of the vectors in the set
- If any other combination gives zero, they are **linearly dependent** and at least one of them is a linear combination of the others

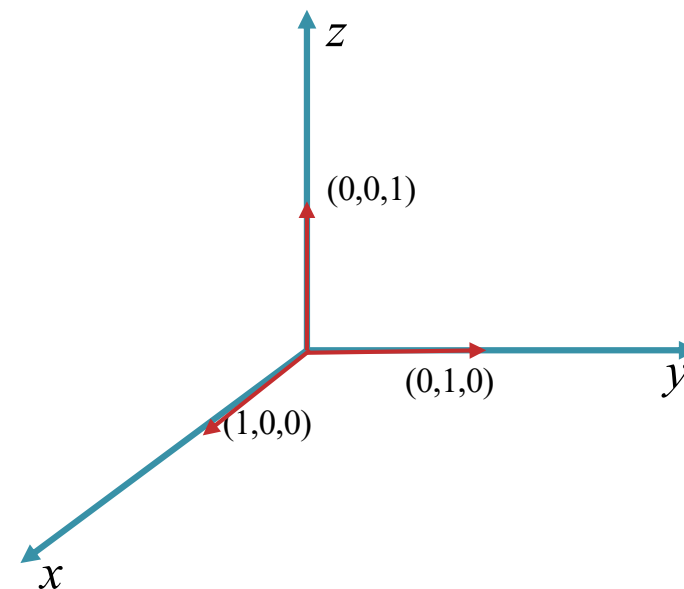
Linear Independence

Example:

- Consider the set $V = \{ [1,0,0]^T, [0,1,0]^T, [0,0,1]^T \}$. Now we represent the zero vector as

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The only values c_1 , c_2 and c_3 can take in the above equation is zero. So, the set V is linearly independent



Linear Dependence & Independence

Example:

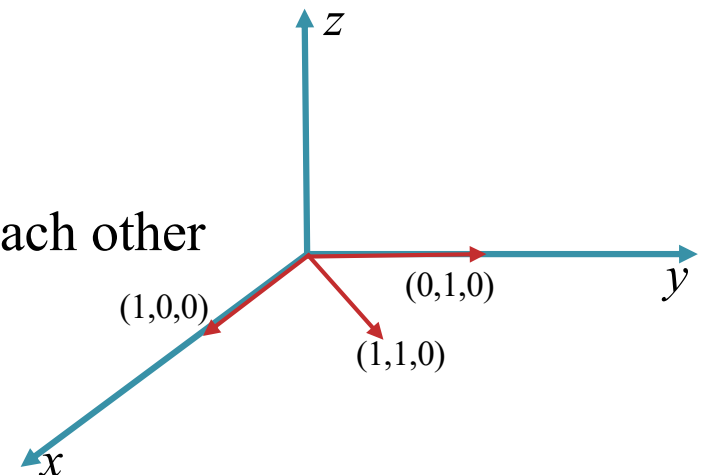
- Consider the set $\mathbf{U} = \{[1,0,0]^T, [0,1,0]^T, [1,1,0]^T\}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, in addition to the solution $c_1 = c_2 = c_3 = 0$, there exists other solutions such as $c_1 = c_2 = -1$ and $c_3 = 1$. So they are linearly dependent on each other. Hence one can be expressed in terms of the rest. For instance, here,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence the vectors are linearly dependent on each other



Linear Dependence & Independence

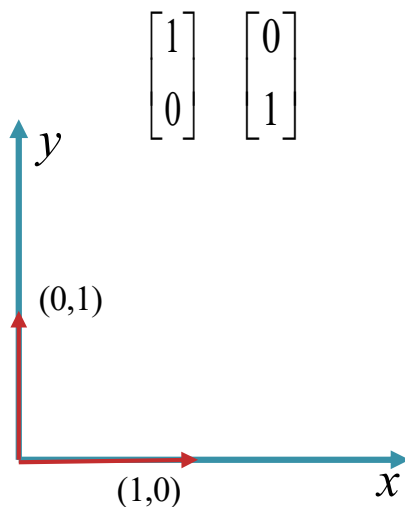
- So the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly dependent
- The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly dependent or independent?
- Answer: Linearly independent
- Zero is the only value that c_1 and c_2 could take in

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

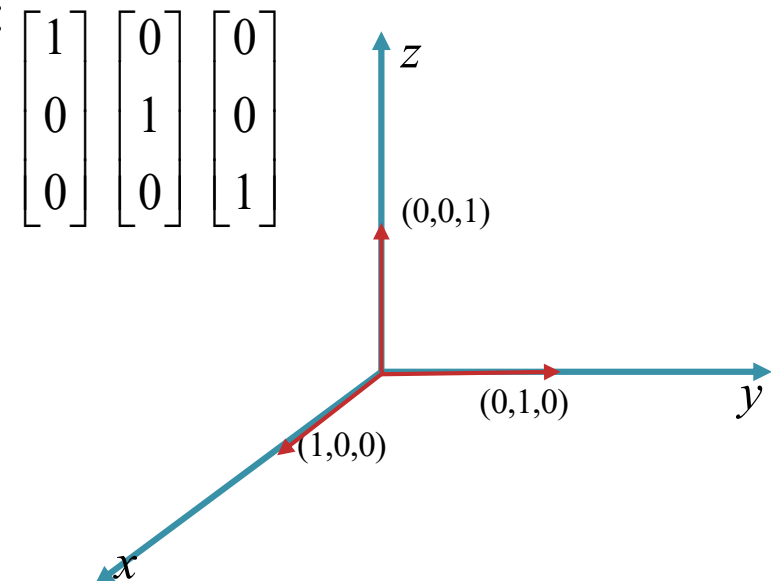
Basis & Dimension of a Vector Space

- **Basis:** A basis for a vector space is a set of vectors having two properties at once:
 - It is linearly independent
 - It spans the space, i.e. a linear combination of the basis vectors can generate any other vector in the vector space
- In more general terms, a basis is a linearly independent spanning set
- Standard basis:

2D:



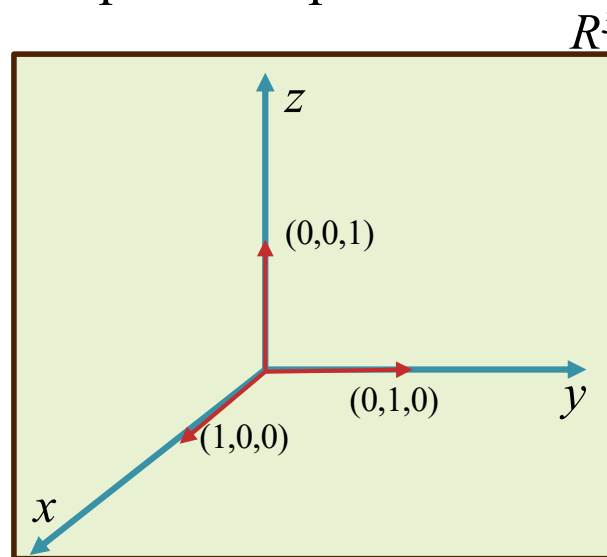
3D:



Basis & Dimension of a Vector Space

- **Span:** Span of a vector set V refers to the vector space generated by all possible linear combinations of vectors present in V

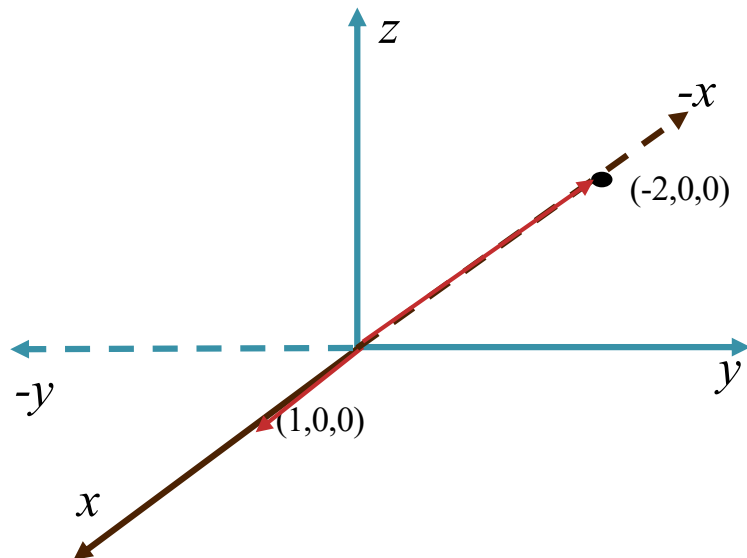
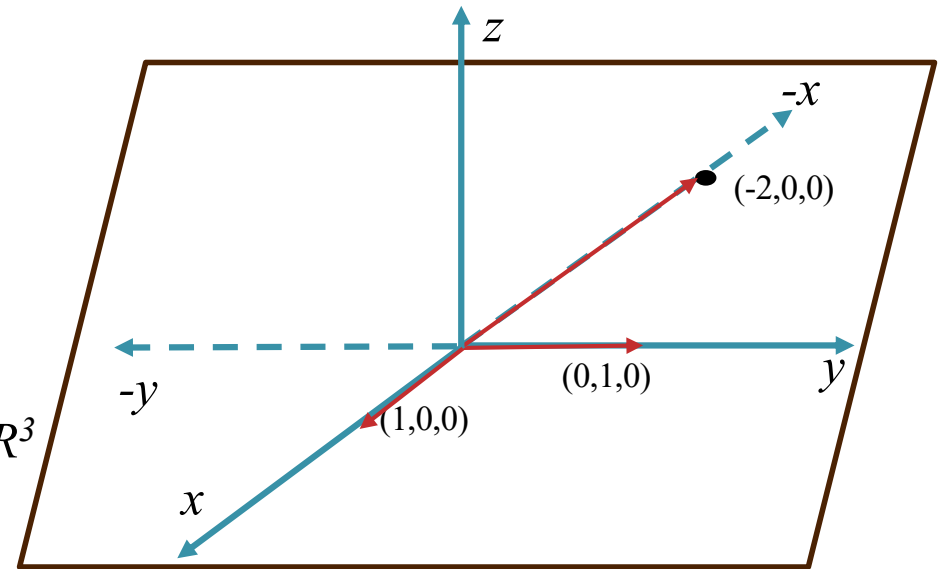
Example: Consider the vector set $V = \{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\}$. The span of V is R^3 . This means that the possible linear combinations of the vectors of V fill the complete R^3 space



- **Dimension:** The maximum number of linearly independent vectors that can be obtained from a vector space is called the **dimension** of that vector space

Basis & Dimension of a Vector Space

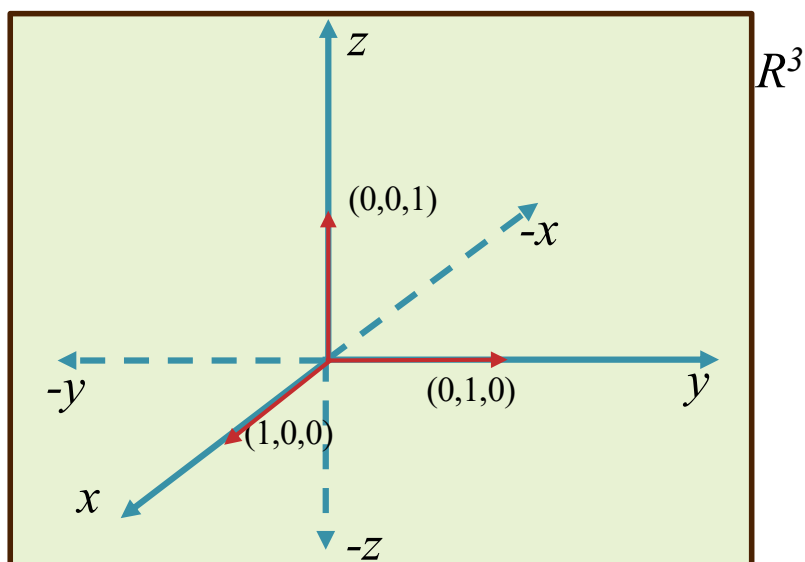
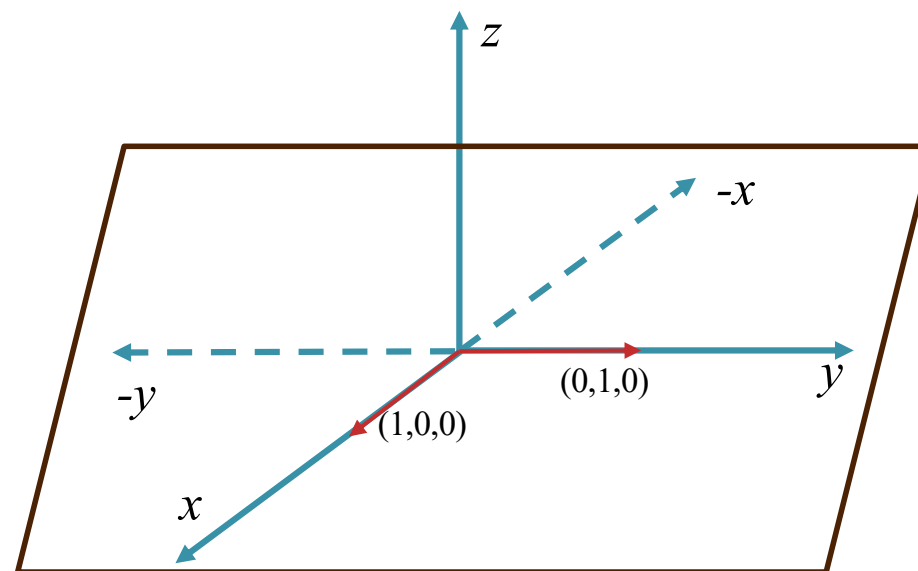
- $V = \{(1,0,0), (0,1,0), (-2,0,0)\}$
- Spans a plane (the x - y plane) in R^3
- The vector $(-2,0,0)$ and $(1,0,0)$ are linearly dependent
- Hence, the vector set V doesn't form a basis in R^3
- **Dimension = 2**



- $V = \{(1,0,0), (-2,0,0)\}$
- Spans only a line
- Linearly dependent
- Hence V doesn't form a basis in R^3
- **Dimension = 1**

Basis & Dimension of a Vector Space

- $V = \{(1,0,0), (0,1,0)\}$
- Spans a plane (the x - y plane) in R^3
- Linearly independent
- Hence, V is a basis for the x - y plane
- ***Dimension = 2***



- $V = \{(1,0,0), (0,1,0), (0,0,1)\}$
- Spans the whole space R^3
- Linearly independent
- So, V is a basis for the space R^3
- ***Dimension = 3***

Rank of a Matrix

- Rank of a matrix refers to the number of linearly independent rows or columns of the matrix: Dimension of column space = Dimension of row space
- It can also be viewed as the number of pivots in Gaussian elimination process

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

Number of linearly independent rows = 2

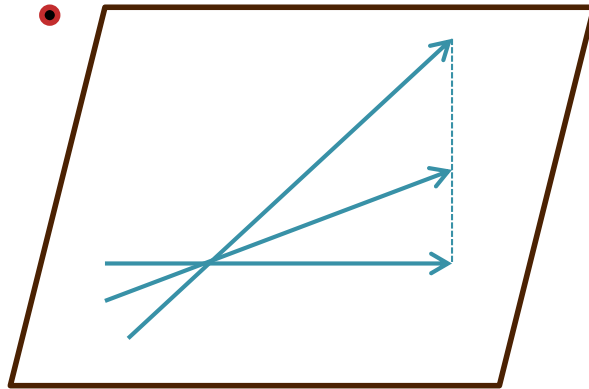
⇒ Rank of the matrix = 2

Existence and Uniqueness of a solution

- $Ax=b$
- When does solution exist?

No solution

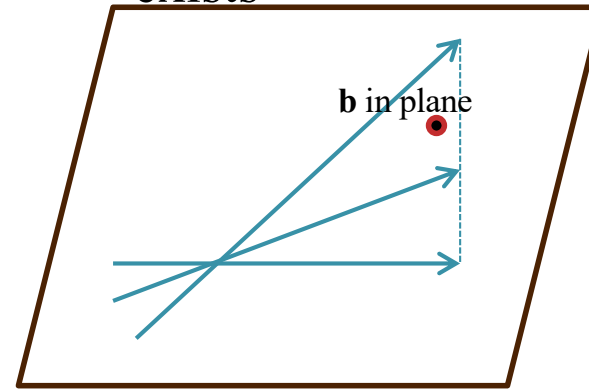
b not in plane



At least one solution exists

✓

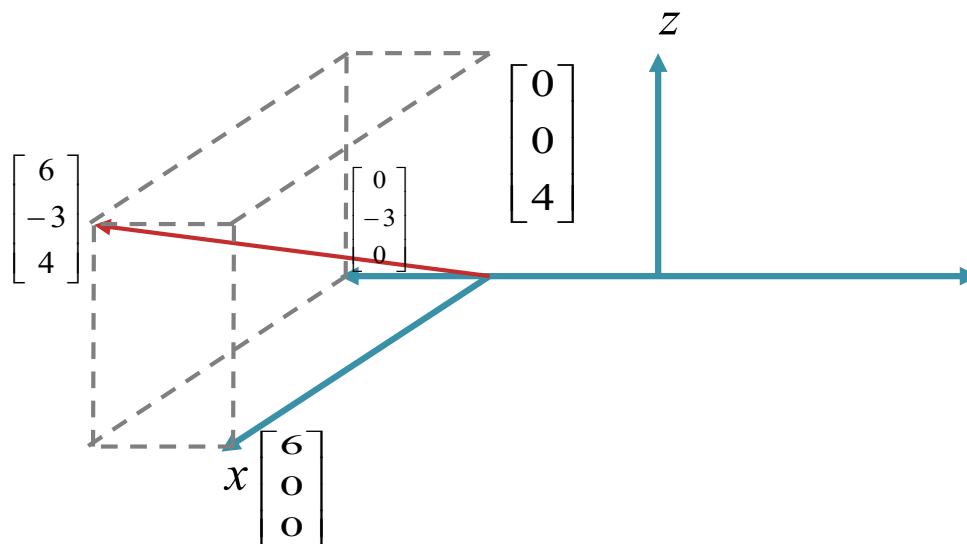
b in plane



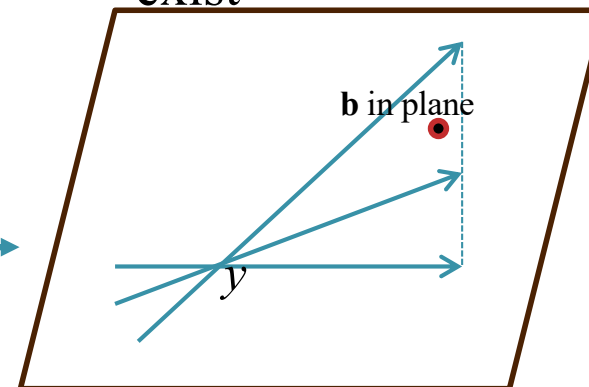
Existence and Uniqueness of a solution

- Uniqueness?

Unique solution



Infinite # solution exist



Null Space

- The null space of a matrix \mathbf{A} consists of all vectors \mathbf{x} such that $\mathbf{Ax} = \mathbf{0}$
- The set of solutions (\mathbf{x} 's) in $\mathbf{Ax} = \mathbf{0}$ is itself a vector space which is called as a null space of \mathbf{A}

$$N(\mathbf{A}) = \{ \mathbf{x} \in R^n / \mathbf{Ax} = \mathbf{0} \}$$

- If the column vectors are linearly independent, the null space contains only the zero vector
- The null space of an invertible matrix contains only zero vector

Null Space

To find null space for the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

- Null space: $N(\mathbf{A}) = \{ \mathbf{x} \in \mathbb{R}^n / \mathbf{A}\mathbf{x} = \mathbf{0} \}$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 3 & 0 \end{array} \right] \xrightarrow{\text{R}_3 = \text{R}_3 - \text{R}_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right]$$
$$\xrightarrow{\text{R}_3 = \text{R}_3 - \text{R}_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 = \text{R}_1 - \text{R}_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Null Space

- We have the equations:

$$x_1 + 2x_3 + x_4 = 0 \Rightarrow x_1 = -2x_3 - x_4$$

$$x_2 + x_4 = 0 \Rightarrow x_2 = -x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- $N(A)$ = scalar multiples of the vectors $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Null Space

- **If the vectors are linearly independent, the null space contains only the zero vector**

- The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent

- What is the null space of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$?

- $\mathbf{Ax} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. What are the values of x_1 and x_2 ?

Null Space

- We have –

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $x_1 = 0$ and $x_2 = 0$ is the only solution that the system of equations could take

⇒ If the vectors are linearly independent, the null space contains only the zero vector

Null Space

- **If the vectors are linearly independent, the null space contains only the zero vector. What about the dependent vectors?**

- Consider two dependent vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

- What is the null space of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$?

- $\mathbf{Ax} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $x_1 = ?$ and $x_2 = ?$

Null Space

- We have –

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} &\Rightarrow \begin{bmatrix} 1 & 2 & | & 0 \\ 2 & 4 & | & 0 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

- We have the equations –

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- We have $N(A) =$ scalar multiples of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- **If the vectors are linearly dependent, the null space also contains non-zero vector(s)**

Null Space: Cause of Non-Uniqueness

- Solutions of $\mathbf{Ax} = \mathbf{b}$ with \mathbf{A} having n columns.
- Solution does not exist if \mathbf{b} does not belong to the column space of \mathbf{A}
- At least one solution exists if \mathbf{b} belongs to the column space of \mathbf{A}
 - Solution unique if null space of \mathbf{A} has only the zero vector
 - Infinite solutions if null space of \mathbf{A} has non-zero vectors

Existence and Uniqueness of a solution

- $\mathbf{Ax} = \mathbf{b}$ has no solution or is inconsistent iff
 $\text{rank}(\mathbf{A}) < \text{rank}(\text{augmented matrix } [\mathbf{A} \ \mathbf{b}])$
[\mathbf{b} does not belong to column space of \mathbf{A}]
- $\mathbf{Ax} = \mathbf{b}$ has a unique solution iff
 $\text{rank}(\mathbf{A}) = \text{rank}(\text{augmented matrix } [\mathbf{A} \ \mathbf{b}]) = n$
[\mathbf{b} belongs to column space of \mathbf{A} and null space of \mathbf{A} has only the zero vector]
- $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions iff
 $\text{rank}(\mathbf{A}) = \text{rank}(\text{augmented matrix } [\mathbf{A} \ \mathbf{b}]) < n$
[\mathbf{b} belongs to column space of \mathbf{A} and null space of \mathbf{A} has non-zero vectors as well]

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 3 \end{bmatrix}$$

- We will include the right hand side as an extra column to \mathbf{A} . That matrix is called as an ‘Augmented matrix’ –

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 1 & 2 & 12 \\ 1 & 0 & 1 & 3 \end{array} \right]$$

- The system is inconsistent as the $\text{rank}(\mathbf{A}) < \text{rank}(\text{augmented matrix})$

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 6 \end{bmatrix}$$

- The system has a unique solution as the $\text{rank}(\mathbf{A}) = \text{rank}(\text{augmented matrix}) = n$

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix}$$

- The system has infinitely many solutions as the $\text{rank}(\mathbf{A}) = \text{rank}(\text{augmented matrix}) < n$

Determinants

- ❑ Determinant is the value associated with the square matrix (matrix with same number of row and columns)
- ❑ Used to determine whether a $n \times n$ matrix has rank n or not
- ❑ Equivalent to finding if inverse of a square matrix exists or not
- ❑ Used for computing eigenvalues as well

Determinants

How to calculate the determinant of a square matrix ?

Expand using the first row

$i+j$ is even
assign a
positive sign

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$$

$i+j$ is odd
assign a
negative sign

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

A matrix is said to be singular if the determinant value is zero

Determinants

Expand using the first row

$$A = \begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants

- If the determinant of a matrix is zero, the matrix is singular
 - Rows or columns are dependent
 - Rank $< n$
- Singular matrices are not invertible
- If the determinant of a matrix is non-zero, the matrix is non-singular or invertible
 - Rows and columns are independent
 - Rank $=n$



Thank you

Acknowledgement for slides:

Prof. Sridharakumar Narasimham (IIT Madras)

Prof. Raghunathan Rengaswamy (IIT Madras)